Gravity Gradients: Their Intrinsic Representation and Applications

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Nomenclature

F	=	field strength
g	=	acceleration of gravity
i, j, k	=	unit vectors aligned with x, y, z
x, y, z	=	reference trihedral (z = local vertical; x, y = directions along lines of curvature)
R_1, R_2	=	principal radii of curvature
U	=	potential function of gravity
V	=	matrix defined by Eqs. (5) and (7)
ξ, η, z	=	reference trihedral derived from x, y, z by a rotation about z by an angle φ
φ	=	see preceding definition
ω	=	angular velocity of reference system

Introduction

GRADIENT phenomena have been under study for a long time.[1-3] Heretofore, emphasis has been upon vehicle attitude problems rather than the basic exploration of gradient phenomena upon a planet.[4-6] This note offers a novel method of exploration of gravity and its anomalies upon planets by an appropriate choice of the system of reference in which gradient magnitudes are expressed.

According to classical potential theory,[7] any gravitational field **F** in empty space (i.e., outside of masses) is solenoidal and irrotational. Denoting its scalar potential by $U(x, y, z)$, we may write $\nabla U = \mathbf{F}$, $\nabla^2 U = 0$, $\nabla \times \mathbf{F} = 0$. It follows from the harmonic nature of U and ∇U that any conceivable field except the zero field is nonuniform in space.

The nonuniformity of the field is twofold. One pertains to the variation of its strength along its own direction, whereas the other appears as a variation of its direction, if there is passage from a line of force to an adjacent one (the "curvature effect").

These considerations offer important possiblities for investigation, provided that the field itself in the (much weaker) gradient environment is eliminated. The first method devised for this elimination goes back to Von Eotvos,[8] who measured horizontal gradients with a torsion pendulum, made insensitive to the influence of gravity by confining its motion to a horizontal plane. The principle underlying this method (based on the curvature and the differential curvature of the equipotential) lends itself to the exploration of gravity and its anomalies from an orbiting vehicle.

Systems of Reference in Gradient Techniques

The term "gravity gradient" denotes the nine second-partial space derivatives of the potential. Of these nine derivatives, five are independent. The irrotational and solenoidal nature of the field imposes four equations ($U_{xy} = U_{yx}$, etc., and $U_{xx} + U_{yy} + U_{zz} = 0$), which reduce the nine gradient terms to five independent components of an absolute second-rank covariant tensor.

Consider $U(x, y, z)$, and suppose that the x, y, z coordinates are chosen in a way locally to make $U_x = U_y = 0$. The directions $x = 0$ and $y = 0$ are those of principal curvature of the equipotential, R_1 and R_2 being the radii of curvature in the x and y directions. In general, $R_1 \neq R_2$. As the field is solenoidal, we have $\partial(R_1 R_2 U_z)/\partial z = 0$, or

$$U_{zz} = -(R_1^{-1} + R_2^{-1})U_z = (R_1^{-1} + R_2^{-1})g \qquad (1)$$

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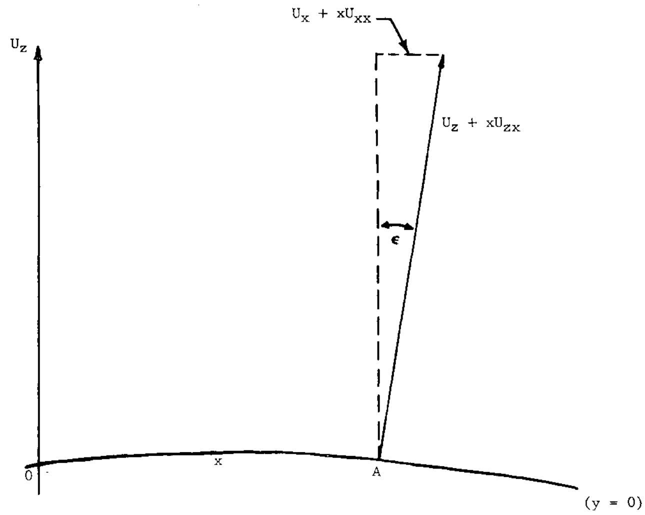

Fig. 1 Variation of the field along a line of curvature $y = 0$ of an equipotential.

Conversely, it is known[9] that the normals drawn upon a surface of constant U along a line of curvature form a developable surface. At a point A upon U = const, at a small distance x (Fig. 1) from 0 along the line of curvature $y = 0$, U_z becomes $U_z + xU_{zx}$. The variation xU_{zx} has no vertical component. Therefore, $U_{zx} = 0$, whereas the U_x component ($U_x = 0$ at $x = 0$) becomes xU_{xx} in A and shows a horizontal increment of $U_z\epsilon$ or $U_z x/R_1$. We have, therefore, a set of simple and easily interpretable relationships ($U_z = -g$) represented by the second gradient matrix of U:

$$\begin{pmatrix} -gR_1^{-1} & 0 & 0 \\ 0 & -gR_2^{-1} & 0 \\ 0 & 0 & g(R_1^{-1} + R_2^{-1}) \end{pmatrix} \qquad (2)$$

If the trihedral of reference has a time-dependent angular velocity ω while aligned at a particular instant with x, y, and z, centrifugal and angular acceleration contributions are to be added. Denoting the position vector within the system of reference (x, y, z) by $\mathbf{s} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, we have a centrifugal field

$$(\boldsymbol{\omega} \times \mathbf{s}) \times \boldsymbol{\omega} = [x(\omega_y^2 + \omega_z^2) - y\omega_x\omega_y - z\omega_x\omega_z]\mathbf{i} + \ldots \qquad (3)$$

and an angular acceleration field

$$\mathbf{s} \times d\boldsymbol{\omega}/dt = (y\dot{\omega}_z - z\dot{\omega}_y)\mathbf{i} + \ldots \qquad (4)$$

Altogether, the composite matrix that includes the contribution of the gradients of the foregoing may be written as

$$||V|| = \begin{pmatrix} \omega_y^2 + \omega_z^2 - gR_1^{-1} & -(\dot{\omega}_z + \omega_x\omega_z) & \dot{\omega}_y - \omega_x\omega_z \\ \dot{\omega}_z - \omega_x\omega_y & \omega_z^2 + \omega_x^2 - gR_2^{-1} & -(\dot{\omega}_x + \omega_y\omega_z) \\ -(\dot{\omega}_y + \omega_x\omega_z) & \dot{\omega}_x - \omega_y\omega_z & \omega_x^2 + \omega_y^2 + g(R_1^{-1} + R_2^{-1}) \end{pmatrix} \qquad (5)$$

A linear combination of the diagonal terms of matrix (5) yields $2\omega^2$. We also have

$$g(R_1^{-1} - R_2^{-1}) = \tfrac{1}{2}(V_{zz} - V_{xx} - V_{yy}) + \omega_z^2 \qquad (6)$$

in which ω_z^2 could be measured separately.

The knowledge of five independent equations represented by the matrix elements V_{xx}, V_{xy}, etc., should yield a solution in gR_1^{-1}, gR_2^{-1}, ω_x, ω_y, ω_z. This operation is theoretically feasible, although it requires a precise alignment and attitude control of the instruments. The gradiometers are both of a "straight" design, such as required for measuring V_{xx}, V_{yy},

V_{zz}, and the "cross" design, needed to sense the off-diagonal terms V_{xy}, V_{yz}, V_{zx}.

Application to Measurements in an Orbiting Vehicle

For an orbiting vehicle, the alignment of z with the local vertical makes the ω_z term drop out from matrix (5). The horizontal gradiometers are supposed to be oriented along ξ and η at an angle φ with respect to the local lines of curvature. Letting $S \equiv \sin$ and $C \equiv \cos$ for brevity, we may write the simplified terms of matrix (5) as

$$\left.\begin{aligned} V_{\xi\xi} &= (\omega_y^2 - gR_1^{-1})C^2\varphi + (\omega_x^2 - gR_2^{-1})S^2\varphi - \omega_x\omega_y S(2\varphi) \\ V_{\xi\eta} &= [g(R_1^{-1} - R_2^{-1}) + \omega_x^2 - \omega_y^2]S\varphi C\varphi - \omega_x\omega_y C(2\varphi) \\ V_{\eta\eta} &= (\omega_y^2 - gR_1^{-1})S^2\varphi + (\omega_x^2 - gR_2^{-1})C^2\varphi + \omega_x\omega_y S(2\varphi) \\ V_{\xi z} &= \dot{\omega}_y C\varphi - \dot{\omega}_x S\varphi \\ V_{\eta z} &= -\dot{\omega}_y S\varphi - \dot{\omega}_x C\varphi \\ V_{zz} &= \omega_x^2 + \omega_y^2 + g(R_1^{-1} + R_2^{-1}) \end{aligned}\right\} \quad (7)$$

Inspection of Eqs. (7) reveals that

$$\left.\begin{aligned} V_{zz} + V_{\xi\xi} + V_{\eta\eta} &= 2\omega^2 \\ V_{zz} - (V_{\xi\xi} + V_{\eta\eta}) &= 2g(R_1^{-1} + R_2^{-1}) \\ V_{\eta\eta} - V_{\xi\xi} &= [\omega_x^2 - \omega_y^2 + g(R_1^{-1} - R_2^{-1})]C(2\varphi) + 2\omega_x\omega_y S(2\varphi) \end{aligned}\right\} \quad (8)$$

Take as a particular example an alignment within the horizontal plane (by gyroscopic means) to make $\omega_\eta = 0$, $\omega_\xi = \omega$. We have

$$\left.\begin{aligned} g(R_1^{-1} - R_2^{-2})C(2\varphi) &= V_{\eta\eta} - V_{\xi\xi} - \omega^2 = \tfrac{1}{2}(V_{\eta\eta} - 3V_{\xi\xi} - V_{zz}) \\ g(R_1^{-1} - R_2^{-1})S(2\varphi) &= 2V_{\xi\eta} \end{aligned}\right\} \quad (9)$$

These equations yield $g(R_1^{-1} - R_2^{-1})$ and the orientation φ of ω with respect to the lines of curvature of the equipotential. V_{zz} should also provide the mean curvature term $g(R_1^{-1} + R_2^{-1})$.

Conclusion

Gravity gradient data can be interpreted directly and related to the geometry of the equipotential. This interpretation is also applicable in a system with a continuous rotation in space. The absolute angular velocity of the reference system and the mean curvature are offered directly. The evaluation of the differential curvature requires, however, an inertial aid, although it might be determined theoretically by gradient measurements alone. Various practical schemes can be selected to explore the potential function of gravity and its local anomalies, using a cross gradiometer and a number of straight ones, depending on the nature of the exploration planned.

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Twin-Gyro Attitude Control Systems

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Several recent investigations[1–3] have considered gyroscopic stabilization of satellites, emphasizing primarily the preservation of a given configuration. Kennedy,[4,5] Havill and Ratcliff,[6] and others have suggested that gyroscopic momentum exchange devices might also be used for attitude control, emphasizing the modification of a given configuration. Specificially, they have suggested that a "twin-gyro" system (two identical but oppositely spinning gyros) has advantages over other momentum exchange devices. This note further examines these advantages and the utility of this device for satellite control.

Configuration

The system to be studied is represented in Fig. 1. The axes X_i ($i = 1, 2, 3$) are centroidal principal inertia axes of the satellite S, with Q representing the mass center. The controlling device consists of two identical disk gyros A and A' with centers at a and a' and oriented such that the axes of each make the same turning angle α with X_2. The turning axes are parallel to X_3, and the rotations of A and A' relative to their own axes are equal in magnitude but opposite in sense. It is sometimes convenient to think of S as also containing two other similar devices symmetrically located with respect to Q with gyro centers on the axes X_3 and X_1 and with turning angle axes, respectively, parallel to X_1 and X_2.

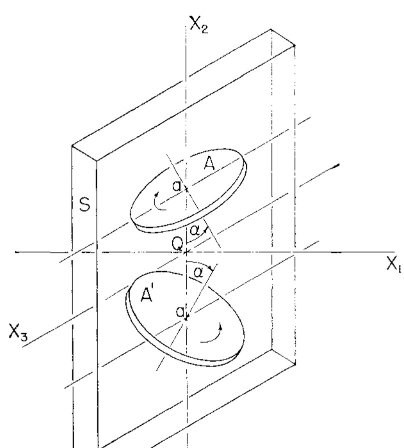

Fig. 1 Satellite with twin-gyro control device.

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